A geometric view of QED:

3. CAR algebras and Clifford algebras as C^*-algebras and their Fock representations

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It has become obvious during the last twenty five years that the representation theory of various infinite dimensional Lie groups and Lie algebras contributes in an essential way to the understanding of quantum field theory. Important examples are the Virasoro algebra in the context of conformal field theories or the affine Kac-Moody algebras.

In particular, the mathematical description of QED is closely related to the representation theory of the infinite dimensional (restricted) unitary group and its central extension by U(1) as has been explained in section 0. To get a better understanding how such a central extension can occur in general quantum theories we have discussed the quantization of symmetries in section 1. In this context and also to obtain a deeper understanding of the representations of the restricted unitary group we think that it is necessary to know the "classical" algebra in QED which is second quantized and on which the restricted unitary group acts as a symmetry group: The CAR algebra (CAR = Canonical Anticommutation Relations) resp. the closely related Clifford algebra.

In this section we give an introduction to the CAR algebras and their Fock representations (second quantizations) by first studying the Clifford algebras and their Fock representations.

C^*-algebras Let us recall some notions:

A *Banach algebra* B (over $\mathbb{C}$) is an associative algebra B over $\mathbb{C}$ with a norm $\| \ \|$ such that B is a complete normed space and the multiplication satisfies

$$\|ab\| \leq \|a\| \, \|b\| \tag{3.1}$$

for all $a, b \in B$.

A *unital Banach algebra* is a Banach algebra B with unit $1 \in B$ whose norm is 1.

A *-*algebra* is an associative algebra A over $\mathbb{C}$ with a *star-involution* $^* : A \longrightarrow A$, $a \mapsto a^*$, i.e. * is a $\mathbb{C}$-antilinear map with $(a^*)^* = a$ for all $a \in A$, satisfying

$$(ab)^* = b^* a^* \tag{3.2}$$

for all $a, b \in A$. By definition, * is an antiautomorphism and an involution of the complex algebra A. A homomorphism between *-algebras A, B (also called a star homomorphism) is an algebra homomorphism – i.e. a complex linear map $h : A \longrightarrow B$ with $h(aa') = h(a)h(a')$ for all $a, a' \in A$ (and $H(1) = 1$ if we have unital algebras) – which respects the involutions:

$$h(a)^* = h(a^*) \tag{3.3}$$

for all $a \in A$.

A C^*-*algebra* is a unital Banach algebra which is a *-algebra as well such that the norm structure is compatible not only with the algebra structure (see 3.1) but also with the involution *:

$$\|a^*a\| = \|a\|^2 \tag{3.4}$$

for all $a \in B$. A (C^*-)homomorphism between C^*-algebras A, B is a continuous star homomorphism.

By definition, in a C^*-algebra one has $\|a\| = \sqrt{\|a^*a\|} = \|a^*\|$. Hence, the involution * is bounded (and therefore continuous) with norm 1. Similarly one can see that every star homomorphism $h : A \longrightarrow B$ between C^*-algebras A, B is automatically continuous with norm ≤ 1: h is *contractive.*

A typical C^*-algebra is given by the algebra $\mathrm{B}(\mathcal{H})$ of all bounded (and complex linear) endomorphisms on a complex Hilbert space $\mathcal{H}$ with the involution given by the adjoint T^* of $T \in \mathrm{B}(\mathcal{H})$. Each C^*-algebra B can be realized as a norm-complete and *-closed subalgebra of such a full endomorphism algebra $\mathrm{B}(\mathcal{H})$ for a suitable choice of $\mathcal{H}$.

Another interesting class of C^*-algebras is given by the algebras $\mathcal{C}(K)$ of complex-valued continuous functions on a compact topological space K.

Given any real Banach algebra A the complexification $A_{\mathbb{C}} := A \otimes_{\mathbb{R}} \mathbb{C}$ is in a natural way a C^*-algebra by defining

$$(a \otimes \lambda)^* := a \otimes \bar{\lambda}$$

for $a \in A$ and $\lambda \in \mathbb{C}$.

The Clifford algebra Let V be a real vector space with a positive definite scalar product $g = (\ ,\ )$, i.e. a euclidean space $V = (V, g)$. For a C^*-algebra B a *Clifford map* is a (real-) linear map $c : V \longrightarrow B$ satisfying

$$c(u)c(v) + c(v)c(u) = 2(u, v)1 \tag{3.5}$$

for all $u, v \in V$. Equivalently, one can require $c(v)^2 = \|v\|^2 1$.

Definition 3.1 (Clifford algebra) *A (complex C^*-) Clifford algebra over the euclidean space $V = (V, g)$ is a C^*-algebra C together with a Clifford map $\iota : V \longrightarrow C$ with the following universal property: Each Clifford map $c : V \longrightarrow B$ into a C^*-algebra B factors uniquely through a homomorphism $h : C \longrightarrow B$, i.e. $c = h \circ \iota$.*

Of course, a homomorphism in this context is a C^*-homomorphism. Evidently, such a Clifford algebra is unique up to isomorphisms (of C^*-algebras).

Remark 3.2 (Variants) *The definition has several variants.*

1. *Instead of $c(u)c(v) + c(v)c(u) = 2(u, v)1$ (in 3.5) we find also $c(u)c(v) + c(v)c(u) = (u, v)1$. The first condition is equivalent to $c(v)c(v) = \|v\|^2 1$, while the second resembles the anticommutation condition for CAR-algebras (see below). The original condition introduced by Clifford is $c(v)c(v) = -\|v\|^2 1$ which is still used in many mathematical articles and books. All these requirements lead to equivalent Clifford algebras.*

2. *A different Clifford algebra (and theory) is given if in the definition 3.1 the Clifford algebra itself is supposed to be only an algebra over $\mathbb{R}$ with unit and the universal property is required for all algebras over $\mathbb{R}$ with unit.*

3. *Again different Clifford algebras arise if in the definition 3.1 the Clifford algebra itself is supposed to be only a complex algebra with unit and the universal property is required for all complex algebras with unit.*

4. *Again different Clifford algebras seem to arise if in the definition 3.1 the Clifford algebra itself is supposed to be only a unital complex *-algebra and the universal property is required for all unital complex *-algebras.*

5. *One can replace the euclidean V with a complex vector space Z equipped with a complex symmetric bilinear form as e.g. in the case of the complexification $V_{\mathbb{C}} = Z$ of a euclidean space $V = (V, g)$ where the bilinear form is the complex-linear extension $g_{\mathbb{C}}$ of g (see below 3.10). In this case one gets the same Clifford algebras as in the real euclidean case as is explained in 3.11.*

6. *Starting with a complex Hilbert space W and requiring the anticommutation relations to match with the Hermitian scalar product (instead of a symmetric complex bilinear form as in 5.) immediately leads to the concept of a CAR algebra (see 3.18 below). We show in this section that the theory of CAR algebras parallels the theory of Clifford algebras.*

7. *Yet another theory appears if the inner product is no longer positive definite but still non degenerate like e.g. a Lorentz inner product. We do not treat this case.*

Theorem 3.3 (Existence of the Clifford algebra) *To each euclidean space $V = (V, g)$ there corresponds a Clifford algebra $\mathcal{C}\ell(V) = \mathcal{C}\ell(V, g)$ unique up to isomorphism.*

In the following we outline a construction of a Clifford algebra over V with the intention to present several natural properties and at the same time explaining some of the consequences in the use of the various different definitions discussed in the remark 3.2.

Since Clifford algebras and CAR algebras are essentially isomorphic we also obtain a construction and description of the CAR algebras (see theorem 3.19).

First of all, we start with the tensor algebra $TV = \bigoplus T^r V$ where $T^r V = V \otimes V \otimes \ldots \otimes V$ is the r-fold tensor product of V with itself. With respect to the tensor multiplication $st := s \otimes t$ for $s, t \in TV$, TV is an associative algebra over $\mathbb{R}$ with unit $1 \in \mathbb{R} = T^0 V$, and V is naturally embedded in TV by $V \cong T^1 V$. Moreover, $T^1 V$ generates TV. Let $I(V)$ be the bilateral ideal in TV generated by $\{u \otimes v + v \otimes u - 2(u, v)1 : u, v \in V\}$. The quotient algebra $C(V) := TV/I(V)$ together with the map $\iota : V \longrightarrow C(V)\,, v \mapsto [v] = v \bmod I(V)$ is a first candidate for a Clifford algebra of V: $C(V)$ is generated bei $\operatorname{im} \iota = \iota(V)$ and to every Clifford map $c : V \longrightarrow B$ into an associative $\mathbb{R}$-algebra B corresponds the unique homomorphism $h : C(V) \longrightarrow B$, induced by $\iota(v) \mapsto c(v) := h(\iota(v))$ for $v \in V$. Clearly, $c(v) = h \circ \iota(v)$.

Thus $C(V)$ turns out to be a Clifford algebra over V with respect to the variant 2 in 3.2.

The universal property implies the following important result:

Theorem 3.4 *If $g : V \longrightarrow V'$ is an isometric $\mathbb{R}$-linear map then there exists a unique algebra homomorphism $\theta_g : C(V) \longrightarrow C(V')$ satisfying*

$$\theta_g \circ \iota = \iota' \circ g\,.$$

Proof: $\iota' \circ g$ is a Clifford map since g preserves the inner products. Hence there is a unique algebra homomorphism $h : C(V) \longrightarrow C(V')$ with $h \circ \iota = \iota' \circ g$ and the result is true for $\theta_g := h$.

□

In the same way one can show that $\theta_{g \circ h} = \theta_g \circ \theta_h$ for isometries g, h. In particular, in the case $V = V'$ and an isometric isomorphism $g : V \longrightarrow V$ (i.e. an *orthogonal map*) the map $\theta_g : C(V) \longrightarrow C(V)$ is an automorphism of algebras (called *Bogoliubov automorphism* related to g) and θ_g defines a representation

$$\theta : \mathrm{O}(V) \longrightarrow \mathrm{Aut}(C(V))\,,\; g \mapsto \theta_g$$

of the *orthogonal group*

$$\mathrm{O}(V) := \{g : V \longrightarrow V \mid (gu, gv) = (u, v) \text{ for all } u, v \in V \text{ and } g \text{ is invertible}\}\,.$$

A Bogoliubov automorphism of particular interest is the one corresponding to the isometric map $-\mathrm{id}_V : V \longrightarrow V$. It is called the *grading operator* und denoted by $\gamma := \theta_{-1}$. We have $\gamma^2 = \mathrm{id}_{C(V)}$:

Lemma 3.5 (Grading operator) *There exists a unique automorphism $\gamma : C(V) \longrightarrow C(V)$ satisfying $\gamma^2 = \mathrm{id}_{C(V)}$ and $\gamma|_V = -\mathrm{id}_V$.*

Here and in the following we often suppress ι in the embedding $\iota : V \longrightarrow C(V)$ and regard it as an inclusion $V \subset C(V)$ in order to get simpler formulas. The condition $\gamma|_V = -\mathrm{id}_V$ has the form $\gamma \circ \iota = -\iota$ when ι is used and emphasized.

An element $\xi \in C(V)$ is called *even* (resp. *odd*) if $\gamma(\xi) = \xi$ (resp. $\gamma(\xi) = -\xi$). $C(V)$ decomposes into even an odd elements: $C(V) = C^+(V) \oplus C^-(V)$ where $C^+(V)$ (resp. $C^-(V)$) is the *even* (resp. *odd*) part of the Clifford algebra $C(V)$. $C^+(V)$ is an algebra whereas $C^-(V)$ is only a vector space and $\oplus$ in this formula denotes the direct sum decomposition of real vector spaces.

Now, $C(V)$ is in general neither a complex algebra nor a *-algebra. Complexifying $C(V)$ gives a $\mathbb{C}$-algebra $C(V)_\mathbb{C} = C(V) \otimes_\mathbb{R} \mathbb{C}$ which can also be understood as the quotient of the complex tensor algebra $T(V_\mathbb{C}) = TV_\mathbb{C}$: $C(V)_\mathbb{C} = TV_\mathbb{C}/I(V)_\mathbb{C}$. $C(V)_\mathbb{C}$ is a unital $\mathbb{C}$-algebra and it serves as a good candidate for a Clifford algebra since it has the universal property with respect to all $\mathbb{C}$-algebras and therefore is the Clifford algebra in the sense of variant 3 in the remark 3.2.

In addition, $C(V)_{\mathbb{C}}$ has a natural involution which makes it to a *-algebra. To introduce this *-structure we need another canonical map $C(V) \longrightarrow C(V)$, the *main antiautomorphism* α which is defined as follows. Let $C(V)^{\circ}_{\mathbb{C}}$ be the algebra opposite to $C(V)_{\mathbb{C}}$, i.e. $C(V)^{\circ}_{\mathbb{C}}$ is the algebra with the same underlying vector space as $C(V)_{\mathbb{C}}$ but with the opposite multiplication: ab in $C(V)^{\circ}_{\mathbb{C}}$ is defined to be ba in $C(V)_{\mathbb{C}}$. Then the identity is an antiautomorphism $C(V)_{\mathbb{C}} \longrightarrow C(V)^{\circ}_{\mathbb{C}}$. Of course, $\mathrm{id}_{C(V)_{\mathbb{C}}} \circ \iota : V \longrightarrow C(V)^{\circ}_{\mathbb{C}}$ is a Clifford map. Hence, there is a unique $\mathbb{C}$-algebra homomorphism $\alpha : C(V)_{\mathbb{C}} \longrightarrow C(V)^{\circ}_{\mathbb{C}}$ with $\mathrm{id}_{C(V)_{\mathbb{C}}} \circ \iota : V \longrightarrow C(V)^{\circ}_{\mathbb{C}} = \alpha \circ \iota$. This means that $\alpha : C(V)_{\mathbb{C}} \longrightarrow C(V)_{\mathbb{C}}$ is an antiautomorphism, i.e. (here) a $\mathbb{C}$-linear isomorphism with $\alpha(ab) = ba$ for all $a, b \in C(V)_{\mathbb{C}}$. Of course, α reverses the tensor products in the following sense: For $s, t \in TV_{\mathbb{C}}$ one has $\alpha([s \otimes t]) = [t \otimes s]$.

Lemma 3.6 (Main antiautomorphism) *There exists a unique antiautomorphism $\alpha : C(V)_{\mathbb{C}} \longrightarrow C(V)_{\mathbb{C}}$ reversing all the tensor products in $TV_{\mathbb{C}}$.*

The involution $^* : C(V)_{\mathbb{C}} \longrightarrow C(V)_{\mathbb{C}}$ is now defined by $a^* := \overline{\alpha(a)}$. Thus, $C(V)_{\mathbb{C}}$ is a *-algebra and satisfies the following universal property. To each Clifford map $c : V \longrightarrow B$ into a *-algebra B with $c(v)^* = c(v)$ (such a map is called *self-adjoint*) for all $v \in V$ there exists a unique star homomorphism $h : C(V)_{\mathbb{C}} \longrightarrow B$ with $c = h \circ \iota$. Hence, $C(V)_{\mathbb{C}}$ is a Clifford algebra over V in the sense of variant 4 in 3.2.

If V is finite dimensional this algebra $C(V)_{\mathbb{C}}$ is also a Banach algebra and a C^*-algebra. $C(V)_{\mathbb{C}}$ is therefore a Clifford algebra over V in the sense of our definiton 3.1.

Example 3.7 *In the case of the euclidean plane V with an orthonormal basis $\{x_1, y_1\}$ a Clifford algebra $C(V)_{\mathbb{C}}$ is the C^*-algebra $\mathbb{C}(2)$ of all 2×2-complex matrices if we set $\iota x_1 = \sigma_2$, $\iota y_1 = \sigma_3$ using the usual Pauli-matrices:*

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example 3.8 (The even dimensional case) *Similarly, in case of a $2n$-dimensional euclidean space V a Clifford algebra $C(V)_{\mathbb{C}}$ is the C^*-algebra $\mathbb{C}(2^n)$ of all $2^n \times 2^n$-complex matrices when we define ι appropriately.*

However, we are not content with the construction of a Clifford algebra for finite dimensional spaces V.

In the infinite dimensional case we need to complete $C(V)_{\mathbb{C}}$ appropriately in order to obtain a C^*-algebra with the universal property of the definition 3.1. If V is separable (and this is the case we are interested in) this can be done using a sequence (V_n) of 2^n-dimensional subspaces $V_n \subset V_{n+1} \subset V$ of V whose union is dense in V and exploiting the corresponding sequence of finite dimensional Clifford algebras $C(V_n)_{\mathbb{C}} \subset C(V_{n+1})_{\mathbb{C}} \subset C(V)_{\mathbb{C}}$ (see, e.g. in the books of Kadison-Ringrose or Pedersen).

We want to follow a different path to complete $C(V)_{\mathbb{C}}$ by using the trace of $C(V)_{\mathbb{C}}$.

Let $\sigma : C(V)_{\mathbb{C}} \longrightarrow \mathbb{C}$ a complex linear functional. σ is called
– *central* if $\sigma(ab) = \sigma(ba)$ for all $a, b \in C(V)_{\mathbb{C}}$,
– *normalized* if $\sigma(1) = 1$ and
– *even* if $\sigma \circ \gamma = \sigma$. ($\gamma$ extends naturally to $C(V)_{\mathbb{C}}$ as a complex-linear automorphism which we denote by the same symbol.)

Lemma 3.9 (Trace) *There exists a unique even, normalized and central complex linear functional $\tau : C(V)_{\mathbb{C}} \longrightarrow \mathbb{C}$ called the trace of $C(V)_{\mathbb{C}}$.*
The trace is a star homomorphism, i.e. $\tau(a^*) = \overline{\tau(a)}$ for all $a \in C(V)_{\mathbb{C}}$.

The existence is first proved for finite dimensional V (in the case of an even dimensional V one can use the example 3.8 by taking the usual trace of matrices). In the infinite dimensional case one uses the fact that $C(V)_{\mathbb{C}}$ can be described as the union of all $C(M)_{\mathbb{C}}$, $M \subset V$ a finite dimensional subspace. The proof of uniqueness requires some lines but is standard.

If now τ is the trace then $\sigma(a) := \overline{\tau(a^*)}$ defines an even, normalized and central complex linear functional, and it follows $\sigma = \tau$ by the first part of the lemma. This establishes that the trace is a star homomorphism.

We now define
$$\langle a, b \rangle_\tau := \tau(a^* b)$$
for $a, b \in C(V)_{\mathbb{C}}$ to obtain a positive definite Hermitian scalar product on $C(V)_{\mathbb{C}}$. (We follow the convention used in the physics literature which requires that the scalar product is complex linear in the second argument and complex antilinear in the first.)

Let $\mathbb{H}_\tau$ be the completion of $C(V)_{\mathbb{C}}$ with respect to this scalar product. Every $a \in C(V)_{\mathbb{C}}$ acts on $C(V)_{\mathbb{C}}$ by multiplication $b \mapsto ab := l(a)b$ (left representation) and $a \mapsto l(a)$ is involution preserving:
$$\langle l(a)\xi, \eta \rangle_\tau = \tau((a\xi)^*\eta) = \tau(\xi^* a^* \eta) = \langle \xi, a^*\eta \rangle_\tau = \langle \xi, l(a^*)\eta \rangle_\tau$$
Hence $l(a)^* = l(a^*)$. One can now deduce that $l(a)$ is bounded with respect to $\langle\, ,\, \rangle_\tau$. A direct proof goes as follows: Since the unitary elements of $C(V)_{\mathbb{C}}$ generate $C(V)_{\mathbb{C}}$ it is enough to restrict to unitary a, i.e. $a^{-1} = a^*$ or $a^* a = 1$:
$$\|l(a)b\|^2 = \langle ab, ab \rangle_\tau = \tau((ab)^* ab) = \tau(b^* a^* ab) = \tau(b^* b) = \langle b, b \rangle_\tau = \|b\|^2 ,$$
hence the operator norm $\|l(a)\|$ of $l(a)$ is 1.

As a result, for each $a \in C(V)_{\mathbb{C}}$ the algebra homomorphism $l(a) : C(V)_{\mathbb{C}} \longrightarrow C(V)_{\mathbb{C}}$ can be continued to a bounded linear map $\mathbb{H}_\tau \longrightarrow \mathbb{H}_\tau$ which we denote by the same symbol $l(a)$. Therefore $C(V)_{\mathbb{C}}$ is represented on the complex Hilbert space $\mathbb{H}_\tau$ by bounded linear functions, the representation being an injective *-algebra homomorphism
$$l : C(V)_{\mathbb{C}} \longrightarrow \mathrm{B}(\mathbb{H}_\tau)$$
into the C^*-algebra $\mathrm{B}(\mathbb{H}_\tau)$.

The completion of $C(V)_{\mathbb{C}}$ with respect to the induced norm is now the Clifford algebra $\mathcal{C}\ell(V)$ we wanted to describe. It has the required universal property and it is a C^*-algebra, since it is isomorphic to the closure of $l(C(V)_{\mathbb{C}})$ in $\mathrm{B}(\mathbb{H}_\tau)$.

In particular, we have (by continuation) the *left regular representation* of the final Clifford algebra $\mathcal{C}\ell(V)$ over the euclidean space V:

$$l : \mathcal{C}\ell(V) \longrightarrow \mathrm{B}(\mathbb{H}_\tau) \ \text{ with } \ l\left(\mathcal{C}\ell(V)\right) = \overline{l\left(C(V)_{\mathbb{C}}\right)}$$

$\square$

We have now established the existence of a Clifford algebra $\mathcal{C}\ell(V)$ over V together with the left regular representation. We should mention that the results which we explained for $C(V)$ and $C(V)_{\mathbb{C}}$ are valid also for this Clifford algebra: We have the grading operator $\gamma : \mathcal{C}\ell(V) \longrightarrow \mathcal{C}\ell(V)$ and the main antiautomorphism α. Similarly, there is the trace and to each $g \in \mathrm{O}(V)$ we obtain the Bogoliubov automorphism

$$\theta_g : \mathcal{C}\ell(V) \rightarrow \mathcal{C}\ell(V)$$

defining a representation of the orthogonal group $\mathrm{O}(V)$

$$\theta : \mathrm{O}(V) \longrightarrow \mathrm{Aut}\, \mathcal{C}\ell(V)\,.$$

In particular, each θ_g is a bounded $\mathbb{C}$-linear star homomorphism and the group homomorphism θ is continuous with respect to the natural norm topology on the orthogonal group $\mathrm{O}(V)$ and the norm topology on the Clifford algebra $\mathcal{C}\ell(V)$.

We observe that there is basically no difference whether the euclidean space V is complete or not. The Clifford algebra $\mathcal{C}\ell(V)$ over V is a Clifford algebra over the completion $\hat{V}$ as well. And the embedding $\iota : V \longrightarrow \mathcal{C}\ell(V)$ has a natural continuation $\hat{\iota} : \hat{V} \longrightarrow \mathcal{C}\ell(V)$.

Hence it is reasonable to assume that V is complete in the following.

Conjugation and complexification A key structure in the study of representations of CAR and Clifford algebras and its application to QED is the conjugation of a complex Hilbert space. We treat the conjugation in the context of the complexification of a euclidean space and show later the connections to complex structures.

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[Dieser Unterabschnitt 'Conjugation and complexification' (wie auch der in Kürze folgende zu komplexen Strukturen) ist ein wenig elementarer und ausführlicher als die anderen, weil 'Konjugationen im Hilbertraum' in der einschlägigen Literatur stiefmütterlich behandelt werden, und nicht alles, was wir brauchen können, leicht bzw. überhaupt zu finden ist. Das liegt möglicherweise daran, dass diese Betrachtungen in ein Standardbuch über Hilberträume thematisch nicht hineinpassen. Tatsächlich sind die folgenden Ausführungen dieses Unterabschnitts eher durch geometrische Vorstellungen geprägt, und meine Darstellung lehnt sich an die Grundlagen von (fast-) komplexen Mannigfaltigkeiten und hermiteschen Vektorbündeln an. Auch wenn das alles ziemlich trivial aussieht, so scheint es in dieser Form nirgends dargestellt zu sein.

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Let us begin with an explicit description of the Hermitian structure of a complex Hilbert space $\mathcal{H}$ with its Hermitian scalar product $h = \langle\, , \rangle$ in relation to its underlying euclidean and symplectic structures. h has a decomposition $h = g + i\omega$ into its real and imaginary parts: $g = \mathrm{Re}\, h = \frac{1}{2}(h + \overline{h})$ and $\omega := \mathrm{Im}\, h = -\frac{i}{2}(h - \overline{h})$.

Lemma 3.10 *g determines a euclidean structure on $\mathcal{H}$ (more precisely on the underlying real space) while ω defines a symplectic structure.*

g and ω are related to the complex multiplication in the following way

$$\omega(z, z') = g(iz, z') \text{ and } g(z, z') = \omega(z, iz') \tag{3.6}$$

for $z, z' \in \mathcal{H}$. Moreover, g and ω are invariant with respect to the multiplication (as is h): $h(iz, iz') = h(z, z')$ implies

$$g(z, z') = g(iz, iz') \text{ and } \omega(z, z') = \omega(iz, iz')\,. \tag{3.7}$$

The Hermitian structure on $\mathcal{H}$ is determined by the euclidean in the following sense: If $g : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ is an invariant euclidean inner product on a complex vector space $\mathcal{H}$ (as a real vector space) then the definition

$$h(z, z') = g(z, z') + ig(iz, z') \tag{3.8}$$

for $z, z' \in \mathcal{H}$ determines a positive definite Hermitian scalar product h with $\operatorname{Re} h = g$. A corresponding result holds for non-degenerate invariant symplectic forms ω on $\mathcal{H}$: $h(z, z') = \omega(z, iz') + i\omega(z, z')$.

Let now V be a complete euclidean space (over $\mathbb{R}$) with inner product $g = (\ ,\ )$. The complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ will also be denoted by $V_{\mathbb{C}} = V \oplus iV$ since every $z = w \otimes \lambda$, $w \in V$, $\lambda = \xi + i\eta \in \mathbb{C}$ has the unique decomposition $z = x + iy$ with $x = w \otimes \xi, y = w \otimes \eta \in V$ where V is identified with $\{v \otimes 1 : v \in V\} \subset V \otimes_{\mathbb{R}} \mathbb{C}$. The inner product $g : V \times V \longrightarrow \mathbb{R}$ can be extended to a complex-bilinear symmetric $g_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}$ by

$$g_{\mathbb{C}}(u + iv, x + iy) := g(u, x) - g(v, y) + i(g(u, y) + g(v, x)) \tag{3.9}$$

for $u, v, x, y \in V$.

$g_{\mathbb{C}}$ defines a natural positive definite Hermitian scalar product on the complex vector space $V_{\mathbb{C}}$ by

$$\langle u + iv, x + iy\rangle := g_{\mathbb{C}}(u - iv, x + iy) = g(u, x) + g(v, y) + i(g(u, y) - g(v, x)) \tag{3.10}$$

for $u, v, x, y \in V$. $V_{\mathbb{C}}$ with this scalar product $h := \langle\ ,\ \rangle$ is a complex Hilbert space such that $h|_{V \times V} = g$.

Observe that the direct decomposition $V_{\mathbb{C}} = V \oplus iV$ is not orthogonal with respect to this scalar product; but it is an orthogonal decompostion with respect to the euclidean structure $\operatorname{Re} h$ on $V_{\mathbb{C}}$ induced by g.

Every complex Hilbert space $\mathcal{H}$ can be represented as the complexification of a euclidean space $V \subset \mathcal{H}$ in many different and non-canonical ways: Choose an orthonormal (Schauder) basis $(e_j)_{j \in I}$ of the Hilbert space $\mathcal{H}$ and set $V := \overline{\operatorname{span}_{\mathbb{R}}\{e_j : j \in I\}}$. It is easy to see that $V \oplus iV = \mathcal{H}$ as Hilbert spaces.

$V_{\mathbb{C}}$ carries a natural conjugation

$$^{-} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}\,,\ v \otimes \lambda \mapsto v \otimes \overline{\lambda} \tag{3.11}$$

or $\overline{u + iv} = u - iv$ for $u, v \in V$. Using this conjugation, the scalar product has also the description

$$\langle z, z' \rangle = g_{\mathbb{C}}(\overline{z}, z')\,. \tag{3.12}$$

We obtain immediately the following result which is of interest for the investigation of CAR algebras (see the definiton 3.18 below) and which is related to the variant 5. in the remark 3.2.

Remark 3.11 *The complexification $V_{\mathbb{C}}$ is in a natural way embedded into $\mathcal{C}\ell(V)$ by the complex linear extension $\iota_{\mathbb{C}}$ of ι to the complexification of V: $\iota_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow \mathcal{C}\ell(V)$. The anticommutation rules are now described by the complex bilinear and symmetric form $g_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}$:*

$$\iota_{\mathbb{C}}(z)\iota_{\mathbb{C}}(z') + \iota_{\mathbb{C}}(z')\iota_{\mathbb{C}}(z) = 2g_{\mathbb{C}}(z, z')1 \tag{3.13}$$

$$\iota_{\mathbb{C}}(\overline{z}) = \iota_{\mathbb{C}}(z)^* \tag{3.14}$$

for all $z, z' \in V_{\mathbb{C}}$.

Definition 3.12 *A conjugation $\Sigma : \mathcal{H} \longrightarrow \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ is an antiunitary map with $\Sigma^2 = \mathrm{id}_{\mathcal{H}}$.*

Here, an antiunitary map is a complex-antilinear map with $\langle \Sigma(z), \Sigma(z') \rangle = \overline{\langle z, z' \rangle}$ for $z, z' \in \mathcal{H}$.

Example 3.13 *The complexification of a euclidean space V comes with a natural conjugation, see 3.11.*

Example 3.14 *Charge conjugation as e.g. in Thaller is a conjugation. Here, $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^m)$ and $\Sigma(\psi) := \mathrm{U}_C(\overline{\psi})$, $\psi \in \mathcal{H}$, where $\mathrm{U}_C : \mathcal{H} \longrightarrow \mathcal{H}$ is a suitable unitary operator.*

Example 3.15 *Let $(e_j)_{j \in I}$ be an orthonormal (Schauder) basis of the Hilbert space $\mathcal{H}$. Define $\Sigma(\sum_{j \in I} \lambda^j e_j) := \sum_{j \in I} \overline{\lambda^j} e_j\,,\ \lambda^j \in \mathbb{C}$. Then Σ is a conjugation (and each conjugation is of this form, see 3.15 below).*

Example 3.16 *Let W be a complex Hilbert space and let $\overline{W}$ be the conjugated Hilbert space (cf. section 0). As a set and a real vector space $\overline{W}$ is the same as W, but with the scalar multiplication $(\lambda, z) \mapsto \overline{\lambda} z$ and inner product $\langle w, w' \rangle = \overline{\langle w, w' \rangle}$. Then the identity $W \longrightarrow \overline{W}$ is the canonical complex-antilinear isomorphism $\sigma : W \longrightarrow \overline{W}$ preserving the scalar products in the sense of $\langle \sigma(w), \sigma(w') \rangle = \overline{\langle w, w' \rangle}$ also denoted by $\overline{w} = \sigma(w)$. Now, the orthogonal sum $\mathcal{H} := W \oplus \overline{W}$ has the conjugation $\Sigma(w + \overline{w'}) := w' + \overline{w}$ for $w, w' \in W$.*

The last example has an obvious generalization to the case of a complex-antilinear isomorphism $\sigma : W \longrightarrow Z$ of Hilbert spaces preserving the scalar products in the sense of $\langle \sigma(w), \sigma(w') \rangle = \overline{\langle w, w' \rangle}$ for $w, w' \in W$: On $\mathcal{H} := W \oplus \sigma W = W \oplus Z$ we have the conjugation $\Sigma(w + \sigma(w')) = w' + \sigma(w)$ or $\Sigma(w + z) = \sigma^{-1}(z) + \sigma w$ for $w, w' \in W$ and $z \in Z$.

We have seen that a complexification of a complete euclidean space V leads to a natural conjugation on the complex Hilbert space $V_{\mathbb{C}}$: $u + iv \mapsto u - iv$. Conversely, a given conjugation Σ in a complex Hilbert space $\mathcal{H}$ with Hermitian scalar product $\langle \, , \rangle = h$ determines a unique way to represent $\mathcal{H}$ as a complexification of a euclidean space V.

Lemma 3.17 *Let $\mathcal{H}$ be a complex Hilbert space with a conjugation Σ. Then there exists a unique real subspace $V \subset \mathcal{H}$ such that the complexification $V_{\mathbb{C}} = V \oplus iV$ is the Hilbert space $\mathcal{H}$ and the conjugation Σ is the natural conjugation induced by the complexification (cf. 3.11): $\Sigma(u + iv) = u - iv$ for $u, v \in V$.*

Proof Let $h = g + i\omega$ be the Hermitian scalar product of $\mathcal{H}$ with $g = \mathrm{Re}\, h$ and $\omega = \mathrm{Im} h$ as above. The real subspace $V := \mathrm{Fix}\, \Sigma = \{ z \in \mathcal{H} : \Sigma(z) = z \}$ is euclidean with respect to $g|_{V \times V}$ (cf. lemma 3.10). Because of $z = \frac{1}{2}(z + \Sigma(z)) + i\frac{i}{2}(\Sigma(z) - z)$ and $\frac{1}{2}(z + \Sigma(z))\, , \frac{i}{2}(\Sigma(z) - z) \in V$ for $z \in \mathcal{H}$, we have $V_{\mathbb{C}} = V + iV = \mathcal{H}$ as complex vector spaces. V is isotropic with respect to the symplectic form ω: For $u, v \in V$ we have $h(u, v) = h(\Sigma(u), \Sigma(v)) = \overline{h(u, v)}$, hence $h(u, v)$ is real and $\omega(u, v) = 0$. Now, the complex bilinear extension $g_{\mathbb{C}}$ of $g|_{V \times V}$ is $g_{\mathbb{C}}(u + iv, x + iy) = \mathrm{Re}(h(u, x) - h(v, y)) + i\,(\mathrm{Re}(h(u, y) + h(v, x))) = h(u, x) - h(v, y) + i\,((h(u, y) + h(v, x))) = h(u - iv, x + iy)$. Hence, $h(z, z') = g_{\mathbb{C}}(\overline{z}, z')$ which shows that the two spaces also agree as Hilbert spaces. Moreover, the conjugation Σ is the natural conjugation $^{-}: V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ because of $\Sigma(u + iv) = \Sigma(u) - i\Sigma(v) = u - iv = \overline{u + iv}$ for $u, v \in V = \mathrm{Fix}\, \Sigma$.

□

In particular, we conclude that every conjugation can be represented as the natural conjugation of a unique complexification. Thus the collection of conjugations on $\mathcal{H}$ is in one-to-one correspondence with the set of euclidean subspaces $V \subset \mathcal{H}$ such that $V_{\mathbb{C}} = \mathcal{H}$ as Hilbert spaces.

Moreover, a conjugation Σ can be described by a suitable basis as in example 3.15: If $(e_j)_{j \in I}$ is a basis of $V = \mathrm{Fix}\, \Sigma$ over $\mathbb{R}$ then $(e_j)_{j \in I}$ is a basis of $V_{\mathbb{C}} = \mathcal{H}$ over $\mathbb{C}$ and

$$\Sigma(\sum_{j \in I} \lambda^j e_j) := \sum_{j \in I} \overline{\lambda^j} e_j\, , \; \lambda^j \in \mathbb{C}\, . \tag{3.15}$$

The CAR algebra Let W be a complex Hilbert space with positive definite Hermitian scalar product $\langle \, , \rangle$ (being complex antilinear in the first argument). A CAR map into a C^*-algebra B is a complex antilinear map $a : W \longrightarrow B$ satisfying the ***canonical anticommutation relations***

$$a(w)a(w')^* + a(w')^* a(w) = \langle w, w' \rangle \, 1 \tag{3.16}$$

$$a(w)a(w') + a(w')a(w) = 0 \tag{3.17}$$

for all $w, w' \in W$.

Two remarks are in order:

1. If one uses the convention that the scalar product of a Hilbert space is complex linear in the first argument and complex antilinear in the second, then a CAR map a as in 3.16 will be required to be complex linear. But the requirement here is in accordance with the majority of the physics literature.

2. Another way of introducing the canonical anticommutation relations requires two maps a and a^* (complex antilinear resp. complex linear) with

$$a(w)a^*(w') + a^*(w')a(w) = \langle w, w' \rangle \, 1 \tag{3.18}$$

$$a(w)a(w') + a(w')a(w) = 0 \tag{3.19}$$

$$a^*(w)a^*(w') + a^*(w')^*a(w) = 0 \tag{3.20}$$

and then proving $a(w)^* = a^*(w)$.

Definition 3.18 (CAR algebra) *A CAR algebra over the complex Hilbert space W is a C^*-algebra C together with a CAR map $\overline{\iota} : W \longrightarrow C$ with the following universal property: Each CAR map $a : V \longrightarrow B$ into a C^*-algebra B factors through a unique homomorphism $h : C \longrightarrow B$, i.e. $a = h \circ \overline{\iota}$.*

Compare with the definition 3.1 of a Clifford algebra. Again we have uniqueness up to isomorphism.

The similarity of the CAR 3.16 with the complex version of the Clifford relations 3.13 and the euclidean Clifford relations 3.5 are obvious. As a consequence, the Clifford algebras and the CAR algebras are essentially the same.

In fact, let W be a complex Hilbert space. We use the conjugated Hilbert space $\overline{W}$ (see the example **??**) with the complex antilinear isomorphism $\sigma : W \longrightarrow \overline{W}, w \mapsto w$, to define the Hilbert space $\mathcal{H} := W \oplus \overline{W}$. $\mathcal{H}$ carries the natural conjugation $\Sigma : \mathcal{H} \longrightarrow \mathcal{H}$, $w + \sigma(w') \mapsto w' + \sigma(w)$. Hence $V := \{x \in \mathcal{H} \, : \, \Sigma(x) = x\} = \{w + \sigma(w) : w \in W\}$ is a real subspace such that $V_{\mathbb{C}} = \mathcal{H}$ as Hilbert spaces (cf. Lemma 3.17). In particular, with the notation $\overline{w} := \sigma(w)$ the Hermitian scalar product $\langle \, , \, \rangle$ on $\mathcal{H}$ can be described by the euclidean inner product $g = (\, , \,)$ on V by

$$\langle x, y \rangle = g_{\mathbb{C}}\left(\overline{x}, y\right) \tag{3.21}$$

for $x, y \in \mathcal{H}$. For later use we point out that

$$\beta : W \longrightarrow V \, , \; w \mapsto 2^{-\frac{1}{2}}(w + \sigma(w)), \tag{3.22}$$

is a real linear isomorphism which induces on V a complex structure (to be defined below). Define $\overline{\iota} : W \longrightarrow \mathcal{C}\ell(V)$ by

$$\overline{\iota}(w) := 2^{-\frac{1}{2}} \, \iota_{\mathbb{C}}(w)^* \, ,$$

$w \in W$. $\overline{\iota}$ is complex antilinear with a natural antilinear extension to all of $\mathcal{H}$ which we denote by $\overline{\iota}$ as well: $\overline{\iota}(z) := 2^{-\frac{1}{2}} \, \iota_{\mathbb{C}}(z)^* = 2^{-\frac{1}{2}} \, \iota_{\mathbb{C}}(\overline{z})$ (cf. 3.14, $z \in \mathcal{H}$.

Theorem 3.19 *$\overline{\iota} : W \longrightarrow \mathcal{C}\ell(V)$ is a CAR algebra over W.*

Proof In addition to being complex antilinear $\overline{\iota}$ is also a CAR map: $\overline{\iota}(w)\overline{\iota}(w') + \overline{\iota}(w')\overline{\iota}(w) = 2^{-1}(\iota_{\mathbb{C}}(w')\iota_{\mathbb{C}}(w) + \iota_{\mathbb{C}}(w)\iota_{\mathbb{C}}(w'))^* = (g_{\mathbb{C}}(w', w)1)^* = \overline{g_{\mathbb{C}}(w', w)}1$ for all $w, w' \in V_{\mathbb{C}}$ according to 3.13. Because of $\overline{\iota}(w')^* = 2^{-\frac{1}{2}}\,\iota_{\mathbb{C}}(w')^{**} = 2^{-\frac{1}{2}}\,\iota_{\mathbb{C}}(w')$ and $\overline{\iota}(w) = 2^{-\frac{1}{2}}\,\iota_{\mathbb{C}}(\overline{w})$ (see 3.14) one obtains $\overline{\iota}(w)\overline{\iota}(w')^* + \overline{\iota}(w')^*\overline{\iota}(w) = \frac{1}{2}\,(\iota_{\mathbb{C}}(\overline{w})\iota_{\mathbb{C}}(w') + \iota_{\mathbb{C}}(w')\iota_{\mathbb{C}}(\overline{w})) = g_{\mathbb{C}}(\overline{w}, w')1 = \langle w, w'\rangle\,1$ (see 3.21) and moreover $\overline{\iota}(w)\overline{\iota}(w') + \overline{\iota}(w')\overline{\iota}(w) = g_{\mathbb{C}}(\overline{w}, \overline{w'})1 = \left\langle w, \overline{w'}\right\rangle 1 = 0$ for all $w, w' \in W$ since $w \perp \overline{w'}$.

Now let $a : W \longrightarrow B$ a CAR map into a C^*-algebra B. We define a Clifford map $c : V \longrightarrow B$ by $c(v) := 2^{\frac{1}{2}}(a(w)^* + a(w))$ if $v = w + \overline{w} \in V$ (each $v \in V$ has a unique decomposition as $v = w + \overline{w}$ with $w \in W$). Certainly, c is $\mathbb{R}$-linear. Moreover, $c(v)^2 = 2(a(w)^* + a(w))(a(w)^* + a(w)) = 2(a(w)^*a(w)^* + a(w)a(w) + a(w)a(w)^* + a(w)^*a(w)) = 2\,\langle w, w\rangle\,1$ (cf. 3.18ff). Since $\|v\|^2 = \langle w + \overline{w}, w + \overline{w}\rangle = 2\,\langle w, w\rangle$ we deduce $c(v)^2 = \|v\|^2\,1$, i.e. c is indeed a Clifford map inducing a unique homomorphism $h : \mathcal{C}\ell(V) \longrightarrow B$ with $c = h \circ \iota$.

For the complex-linear extension $c_{\mathbb{C}}$ we get $c_{\mathbb{C}} = h \circ \iota_{\mathbb{C}}$. And for $w \in W$ we can write $w = \frac{1}{2}(w + \overline{w}) + i\frac{1}{2}(-iw + \overline{-iw})$ with $w + \overline{w}, -iw + \overline{-iw} \in V$, hence, we obtain $c_{\mathbb{C}}(w) = \frac{1}{2}c(w+\overline{w}) + i\frac{1}{2}c(-iw+\overline{-iw}) = 2^{\frac{1}{2}}\frac{1}{2}(a(w)^* + a(w) + ia(-iw)^* + ia(-iw)) = 2^{\frac{1}{2}}a(w)^*$ since a is complex antilinear on W. Finally, we deduce $a(w) = a(w)^{**} = 2^{-\frac{1}{2}}c_{\mathbb{C}}(w)^* = 2^{-\frac{1}{2}}(h \circ \iota_{\mathbb{C}}(w))^* = h(2^{-\frac{1}{2}}\iota_{\mathbb{C}}(w)^*) = h \circ \overline{\iota}(w)$ since h is a star homomorphism.

□

With $\mathcal{A}(W)$we shall denote the CAR algebra over the Hilbert space W (uniquely determined up to C^*-algebra isomorphisms) which can be identified with the Clifford algebra $\mathcal{C}\ell(V)$ for a suitable euclidean V.

Conversely,
A variant

The complex structure in its different disguises We have encountered a complex structure of the basic euclidean space V already in the last paragraph. As a basis for the proof of Theorem 3.19 we constructed to a given complex Hilbert space another Hilbert space $\mathcal{H} = W \oplus \overline{W} = V_{\mathbb{C}}$ with a canonical real isomorphism $\beta : W \longrightarrow V$ (see 3.22) inducing a complex scalar multiplication (i.e. complex structure) on the real vector space V.

Definition 3.20 (Complex structure) *A complex structure on a real vector space V is a (real) linear map $J : V \longrightarrow V$ such that $J^2 = 1$.*

Defining $iv := Jv$ for $v \in V$ gives indeed a scalar multiplication such that V becomes a complex vector space which we shall denote by V_J in the following. On the other hand for a given complex vector space W the map $J : W \longrightarrow W\,,\ w \mapsto iw$ (or $-iw$) is of course a complex structure with respect to the underlying real vector space.

We are mainly interested in complex structures J on a euclidean vector space V with an inner product $g = (\ ,\ )$ which are orthogonal, i.e. satisfying $(Jv, Jv') = (v, v')$. Such a complex structure defines a positive definite Hermitian scalar product on V_J by

$$\langle v, v' \rangle_J := (v, v') + i\,(Jv, v') \tag{3.23}$$

or

$$h_J(v, v') = g(v, v') + ig(Jv, v')$$

for $v, v' \in V$.

Let $\mathcal{J}(V)$ denote the set of all orthogonal complex structures on the euclidean space $V = (V, g)$. Any two such complex structures J, J' lead to isomorphic and isometric complex Hilbert spaces V_J and $V_{J'}$. Each such isometry $g : V_{J'} \longrightarrow V_J$ satisfies $\langle gv, gv' \rangle = (gv, gv') + i\,(Jgv, gv') = (v, v') + i\,(J'v, v')$. Hence, g is orthogonal with respect to the original inner product $(\ ,\ )$ and $J' = g^{-1}Jg$. As a consequence, the orthogonal group $\mathrm{O}(V)$ acts on $\mathcal{J}(V)$ by $J \mapsto g^{-1}Jg$. The isotropy group of this action at a point $J \in \mathcal{J}(V)$ is the unitary group $\mathrm{U}(V_J)$. The space of inequivalent orthogonal complex structures is therefore the homogeneous space $\mathrm{O}(V)/\mathrm{U}(V_J)$ which is similar in many aspects to the grassmannian as we will see later.

In order to describe different and inequivalent representations of the CAR and Clifford algebras we are interested in orthogonal decompositions $V_{\mathbb{C}} = W \oplus Z$ of the complexification of the complete euclidean space $V = (V, g)$ and relate them to complex structures on V. The Hilbert space $V_{\mathbb{C}}$ has a canonical conjugation (cf. 3.11) $\Sigma : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}\,,\ v \otimes \lambda \mapsto v \otimes \overline{\lambda} = \Sigma(v \otimes \lambda) := \overline{v \otimes \lambda}$ with $V = \{w \in V_{\mathbb{C}} : \Sigma(w) = w\,\}$.

Lemma 3.21 *Let $V_{\mathbb{C}} = W + Z$ be a direct decomposition into closed subspaces, i.e. $W \cap Z = \emptyset$. Then any two of the following three conditions*

1. *W and Z are conjugate, i.e. $\overline{W} = Z$ (with respect to the natural conjugation induced on $V_{\mathbb{C}}$)*
2. *W and Z are orthogonal: $V_{\mathbb{C}} = W \oplus Z$*
3. *W and Z are isotropic (with respect to $g_{\mathbb{C}}$)*

imply the third.

Proof

2.,3. $\Rightarrow$ 1.

We have to show: $w \in W \Rightarrow \overline{w} \in Z$. $w \in W$ has the unique decomposition $w = x + y$ with $x \in W, y \in Z$. $\|x\|^2 = \langle x, x + y \rangle - \langle x, y \rangle = \langle x, \overline{w} \rangle$ since $\langle x, y \rangle = 0$ by 2. It follows that $\|x\|^2 = g_{\mathbb{C}}(x, w) = 0$ because of 3. (since $x, w \in W$). Consequently, $x = 0$ and $\overline{w} = y \in Z$.

1.,2. $\Rightarrow$ 3.

To show that W is isotropic, we consider $x, y \in W$: $g_{\mathbb{C}}(x, y) = \langle x, \overline{y} \rangle = 0$ since $\overline{y} \in Z$ by 2. and then $x \perp y$.

1.,3. $\Rightarrow$ 2.

Given $x \in W$ and $y \in Z$ we have $\langle x, y \rangle = g_{\mathbb{C}}(x, \overline{y}) = 0$ since $\overline{y} \in W$ and W is isotropic.

$\square$

The situation in the lemma can best be described be writing $V_{\mathbb{C}} = W \oplus \overline{W}$.

Lemma 3.22 (Induced complex structure) *Let $\mathcal{H}$ be a complex Hilbert space with a conjugation Σ which determines a euclidean subspace $V = \mathrm{Fix}\,\Sigma$ with $V_{\mathbb{C}} = \mathcal{H}$. Let $\mathcal{H} = W \oplus \Sigma W$ be an orthogonal decompositon. Then $\beta : W \longrightarrow V$, $\beta(w) := 2^{-\frac{1}{2}}(w + \Sigma w)$, $w \in W$, (see 3.22) is an isometry with respect to the euclidean structures on V and W induced by the Hermitian scalar product h and $Jv := \beta(i\beta^{-1}(v))$, $v \in V$ defines an orthogonal complex structure on V such that $\beta : W \longrightarrow V_J$ becomes a complex-linear isometry.*

Proof In fact, β is not only a real-linear isomorphism but also an isometry since for $w, w' \in W$ we have (denoting $\Sigma(z) = \overline{z}$ as before) $\langle \beta(w), \beta(w') \rangle = 2^{-1} \langle w + \overline{w}, w' + \overline{w'} \rangle = 2^{-1}(\langle w, w' \rangle + \langle \overline{w}, w' \rangle + \langle w, \overline{w'} \rangle + \langle \overline{w}, \overline{w'} \rangle) = 2^{-1}(\langle w, w' \rangle + \overline{\langle w, w' \rangle}) = 2^{-1} 2\mathrm{Re}\, \langle w, w' \rangle = (w, w')$ (W is orthogonal to $\overline{W}$) which implies $(\beta(w), \beta(w')) = (w, w')$. Now, β induces on V an orthogonal complex structure by $Jv := \beta(i\beta^{-1}(v))\,,\ v \in V$: $J^2 = \beta i \beta^{-1} \circ \beta i \beta^{-1} = \beta i^2 \beta^{-1} = -1$ and J is an orthogonal map as a composition of orthogonal mappings. By definition of the new complex structure, $i\beta(w) = J(\beta(w)) = \beta(iw)$, therefore β is complex linear and thus an isometry of Hilbert spaces: $\beta : W \longrightarrow V_J$.

Any complex structure J on V can be continued to a complex linear map $J_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$. $J_{\mathbb{C}}$ is unitary and satisfies ${J_{\mathbb{C}}}^2 = -1$. The two eigenspaces $V^+ := \ker(J_{\mathbb{C}} - i\,\mathrm{id}_{V_{\mathbb{C}}})$ and $V^- := \ker(J_{\mathbb{C}} + i\,\mathrm{id}_{V_{\mathbb{C}}})$ define an orthogonal decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$ where V^+, V^- are isotropic and conjugate to each other: $\overline{V^+} = V^-$.

Conversely, a decompositon $V_{\mathbb{C}} = W + Z$ as in lemma 3.21 and satisfying two and hence all three of the conditions there defines a complex structure J and is induced by J in the following way: Define $J_{\mathbb{C}} = iE_W - iE_Z$ where E_W denotes the orthogonal projection onto W (and E_Z correspondingly the orthogonal projection onto Z, note that $E_W + E_Z = 1$). $J_{\mathbb{C}}$ is unitary with $(J_{\mathbb{C}})^2 = -1$. Then $J := J_{\mathbb{C}}|_V$ maps V into V (since $V = \{ w + \overline{w} \;:\; w \in W \}$ and $J(w + \overline{w}) = iw - i\overline{w} = iw + \overline{iw} \in V$) and J is orthogonal satisfying $J^2 = -1$. Therefore, J is a complex structure on V such that $W = \ker(J_{\mathbb{C}} - i1) = V^+$ and $Z = \ker(J_{\mathbb{C}} + i1) = V^-$. The inverse of β has the following form

$$\beta^{-1}(v) = 2^{-\frac{1}{2}}(v - iJv)\,,\ v \in V\,. \tag{3.24}$$

This follows from $J_{\mathbb{C}}(v - iJv) = Jv + iv = i(v - iJv)$, i.e. $v - iJv \in V^+ = W$ for $v \in V$ and $\beta(2^{-\frac{1}{2}}(v - iJv)) = 2^{-1}(v - iJv + \overline{v - iJv}) = v$.

Note that in the finite dimensional case, V has to be even-dimensional in order to admit a complex structure. In that case the complex structures are parametrized by the homogeneous manifold $\mathrm{O}(2n)/\mathrm{U}(n)$.

Fock space of the Clifford algebra Let V be as before a complete euclidean space and let J be a complex structure on V.

The Hermitian scalar product $\langle\, ,\, \rangle_J$ on the Hilbert space V_J induces on the exterior algebra ΛV_J a Hermitian scalar product such that it is a pre-Hilbert space. The completion is the Fock space $\mathcal{F}(V_J)$.

Define for $v \in V$:

$$d_v(\xi) := v \wedge \xi\, ,\ , \xi \in \Lambda V_J \tag{3.25}$$

$$i_v(v_0 \wedge v_1 \cdots \wedge v_m) = \sum_{j=0}^{j=m} (-1)^j \langle v, v_j \rangle_J \, v_0 \wedge v_1 \cdots \wedge \hat{v}_j \wedge \ldots \wedge v_m \tag{3.26}$$

Lemma 3.23 *d_v and i_v are bounded complex-linear operators satisfying*

$$\|d_v \xi\|^2 + \|i_v \xi\|^2 = \|v\|^2 \|\xi\|^2$$

$$\langle d_v \xi, \eta \rangle_J = \langle \xi, i_v \eta \rangle_J$$

$$d_v \circ d_v = 0\, ,\ i_v \circ i_v = 0$$

$$i_v d_{v'} + d_{v'} i_v = \langle v, v' \rangle$$

for $v, v' \in V$.

Hence, the operators d_v, i_v have continuous complex-linear extensions to $\mathcal{F}(V_J)$ and they are mutually adjoint. Let $a(v) : \mathcal{F}(V_J) \longrightarrow \mathcal{F}(V_J)$ denote the extension of i_v, then $a^*(v) = a(v)^*$ is the extension of d_v. We obtain the canonical anticommutation relations from lemma 3.23:

$$a(v)a^*(v') + a^*(v')a(v) = \langle v, v' \rangle_J 1$$

$$a(v)a(v') + a(v)a(v') = 0 = a^*(v)a^*(v') + a^*(v)a^*(v')$$

Note that $a^* : V_J \longrightarrow \mathrm{B}(\mathcal{F}(V_J))$ is complex-linear and $a : V_J \longrightarrow \mathrm{B}(\mathcal{F}(V_J))$ is complex antilinear.

Theorem 3.24 *$\mathcal{C}\ell(V)$ has a natural representation $\pi_J : \mathcal{C}\ell(V) \longrightarrow \mathrm{B}(\mathcal{F}(V_J))$ on the Fock space $\mathcal{F}(V_J)$ induced by*

$$c(v) := a^*(v) + a(v)\, ,\ v \in V\, .$$

Proof Clearly, $c(v)$ is a bounded complex-linear operator on the Fock space $\mathcal{F}(V_J)$ and $c : V \longrightarrow \mathcal{C}\ell(V)$ is linear over $\mathbb{R}$. Moreover, the canonical anticommutation relations imply $c(v)^2 = (a^*(v) + a(v))(a^*(v) + a(v)) = a^*(v)a^*(v) + a(v)a(v) + a^*(v)a(v) + a(v)a^*(v) = \langle v, v \rangle = (v, v) = \|v\|^2 1$. Hence, c is a Clifford map inducing a homomorphism (of C^*-algebras) $h : \mathcal{C}\ell(V) \longrightarrow \mathrm{B}(\mathcal{F}(V_J))$ with $c = h \circ \iota$. This h is the representation map $\pi_J := h$.

□

Lemma 3.25 *The representation of the theorem is irreducible and any $\Omega \in \Lambda^0 V_J \cong \mathbb{C}$, $\Omega \neq 0$, is cyclic.*

Proof. Evidently, $\operatorname{span}_{\mathbb{C}} \{a^*(v_1)a^*(v_2)\dots a^*(v_m)\Omega \,:\, v_k \in V_J\}$ which is the same as $\operatorname{span}_{\mathbb{C}} \{v_1 \wedge v_2)\dots \wedge v_m \,:\, v_k \in V_J\}$ is dense in $\mathcal{F}(V_J)$, hence Ω is cyclic and π_J is irreducible.

□

Theorem 3.26 *Let J, K be orthogonal complex structures on the complete euclidean space V. Then π_J and π_K are unitarily equivalent if and only if $J - K$ is a Hilbert Schmidt operator.*

As a consequence, in case of $J - K$ being Hilbert Schmidt there is an intertwining unitary operator $T : \mathcal{F}(V_J) \longrightarrow \mathcal{F}(V_K)$ such that $T \circ \pi_J(c) = \pi_K(c) \circ T$ for all $c \in \mathcal{C}\ell(V)$. This is the original result of Shale-Stinespring (cf. Theorem 0.4) and it has to do with the implementation problem: By definition, an operator $g \in \mathrm{O}(V)$ can be *implemented* on $\mathcal{F}(V_J)$ (or in π_J) if there exists a unitary operator $g^\sim : \mathcal{F}(V_J) \longrightarrow \mathcal{F}(V_J)$ such that

$$g^\sim \circ \pi_J(c) = \pi_J \theta_g(c) \circ g^\sim\,, \text{ for all } c \in \mathcal{C}\ell(V)\,. \tag{3.27}$$

Theorem 3.27 *$g \in \mathrm{O}(V)$ has an implementation in π_J if and only if $[g, J]$ is Hilbert Schmidt. If it exists it is unique up to a scalar factor of modulus 1.*

This follows from the observation, that $[g, J]$ is Hilbert Schmidt for $g \in \mathrm{O}(V)$ if and only if $g^{-1}[g, J] = J - g^{-1}Jg$ is Hilbert Schmidt, i.e. iff for the orthogonal complex structure $K = g^{-1}Jg$ the condition of the theorem is fulfilled: $J - K$ is Hilbert Schmidt. Moreover, $\pi_J \circ \theta_g = \pi_K$. Hence the intertwining unitary operator T satisfies $T \circ \pi_J(c) = \pi_J(\theta_g(c)) \circ T$ and is an implementation $g^\sim = T$ of g.

□

Definition 3.28 *The restricted orthogonal group is*

$$\mathrm{O}_{\mathrm{res}}(V, J) = \mathrm{O}_{\mathrm{res}}(V) := \{g \in \mathrm{O}(V) \,:\, [g, J] \text{ Hilbert Schmidt}\}\,.$$

As discussed before, the last theorem induces a projective representation

$$\rho_J : \mathrm{O}_{\mathrm{res}}(V, J) \longrightarrow \mathrm{Aut}\,(\mathbb{P}\mathcal{F}(V_J))$$

which leads to a representation on the Fock space $\mathcal{F}(V_J)$ of a central extension $\mathrm{O}_{\mathrm{res}}{}^\sim(V, J^)$ of the restricted orthogonal group.

ρ_J can be shown to be continuous.

To understand the Hilbert Schmidt condition it is helpful to decompose a given orthogonal map $g \in \mathrm{O}(V)$ into its complex-linear part $C^g := \frac{1}{2}(g - JgJ)$ and its antilinear part $A^g := \frac{1}{2}(g + JgJ)$ with respect to J: $g = C^g + A^g$. Since $C_g J = JC_g$, i.e. $[C_g, J] = 0$, and $A_g J = -JA_g$ we get $[g, J] = 2A_g J$ and deduce:

Lemma 3.29 *$[g, J]$ Hilbert Schmidt $\Longleftrightarrow A_g$ Hilbert Schmidt.*

The $\mathbb{C}$-linear extension $g_{\mathbb{C}}$ of g from V to $V_{\mathbb{C}}$ is unitary with respect to the induced Hermitian scalar product on $V_{\mathbb{C}}$. Regarding the decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$ and writing $g_{\mathbb{C}}$ as a block matrix one gets

$$g_{\mathbb{C}} = \begin{pmatrix} C^g_{\mathbb{C}} & A^g_{\mathbb{C}} \\ A^g_{\mathbb{C}} & C^g_{\mathbb{C}} \end{pmatrix}$$

using the $\mathbb{C}$-linear extensions $C^g_{\mathbb{C}}, A^g_{\mathbb{C}}$ of C_g, A_g.

This observation gives an immediate connection to the discussion of the restricted unitary group (cf. section 0 and the following subsection).

$\mathrm{O}_{\mathrm{res}}(V, J)$ has a natural structure of a real Banach Lie group with respect to the norm $\|g\|_r := \|C_g\| + \|A_g\|_{HS}$. It has two connected components, both of which are simply connected.

Fock space of the CAR algebra For a general complex Hilbert space $\mathcal{H}$ with scalar product $h = \langle\, ,\, \rangle$ we can use the construction of the last subsection to obtain the *wave representation* π of the CAR algebra $\mathcal{A}(\mathcal{H})$ on the Fock space $\mathcal{F}(\mathcal{H})$. Here, $\mathcal{F}(\mathcal{H})$ is (again) the Hilbert space completion of the pre-Hilbert space $\Lambda\mathcal{H}$ with the obvious scalar product and to construct the representation $\pi = \pi_{\mathcal{H}}$ one first considers the complex-linear extension $a^*(z) : \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{H})$ of the operator $d_z : \Lambda\mathcal{H} \longrightarrow \Lambda\mathcal{H}$, $\xi \mapsto z \wedge \xi$, for each $z \in \mathcal{H}$. Thus we have a complex-linear map $a^* : \mathcal{H} \longrightarrow \mathrm{B}(\mathcal{F}(\mathcal{H}))$ with the following anticommutation relations for $a(z) := a^*(z)^*$, $z \in \mathcal{H}$ and a^*:

$$a(z)a^*(z') + a^*(z')a(z) = \langle z, z' \rangle\, 1$$

$$a(z)a(z') + a(z)a(z') = 0 = a^*(z)a^*(z') + a^*(z)a^*(z')$$

By the universal property of the CAR algebra $\mathcal{A}(\mathcal{H})$ there exists a unique homomorphism $\pi : \mathcal{A}(\mathcal{H}) \longrightarrow \mathrm{B}(\mathcal{F}(\mathcal{H}))$ with $a = \pi \circ \bar{\imath}$. π is the wave representation.

To produce other irreducible representations of the CAR algebra $\mathcal{A}(\mathcal{H})$ suitable additional properties of $\mathcal{H}$ are exploited. In the light of the example QED it appears reasonable to require the existence of a conjugation $\Sigma : \mathcal{H} \longrightarrow \mathcal{H}$ as a basic structure of our Hilbert space which is the same as to describe $\mathcal{H}$ as the complexification $V_{\mathbb{C}}$ of a real euclidean subspace $V \subset \mathcal{H}$. In addition, we need an orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (for example given by the eigen spaces of positive resp. negative eigenvalues of the Dirac operator, cf. section 0) respecting the conjugation, i.e. $\Sigma(\mathcal{H}^+) = \mathcal{H}^-$ and consequently $\Sigma(\mathcal{H}^-) = \mathcal{H}^+$ as well. Then $\mathcal{H}^+$ isotropic (cf. lemma 3.21). Such an orthogonal decomposition is given by a self adjoint projection $P = P^+$ satisfying $1_{\mathcal{H}} = P + \Sigma P \Sigma$, or equivalently by an orthogonal complex structure J on V such that $\mathcal{H}^+ = \ker(J_{\mathbb{C}} - i1_{\mathcal{H}})$ and $\mathcal{H}^- = \ker(J_{\mathbb{C}} + i1_{\mathcal{H}})$ (cf. lemma 3.22).

Under these circumstances one regards the identification

$$\mathcal{A}(\mathcal{H}^+ \oplus \mathcal{H}^-) \cong \mathcal{A}(\mathcal{H}^+)\hat{\otimes}\mathcal{A}(\mathcal{H}^-) \cong \mathcal{A}(\mathcal{H}^+)\hat{\otimes}\mathcal{A}(\Sigma\mathcal{H}^-) \cong \mathcal{A}(P\mathcal{H})\hat{\otimes}\mathcal{A}(\Sigma(1-P)\mathcal{H})$$

and mixes the wave representations of $\mathcal{A}(\mathcal{H}^+)$ and $\mathcal{A}(\Sigma\mathcal{H}^-)$ appropriately to obtain the following ansatz:

The Fock space is now the Hilbert space

$$\mathcal{F}_P = \mathcal{F}(\mathcal{H}, \Sigma, P) := \Lambda\mathcal{H}^+\hat{\otimes}\Lambda\Sigma\mathcal{H}^-$$

where the Hermitian scalar product on $\Lambda\mathcal{H}^+ \otimes \Lambda\Sigma\mathcal{H}^-$ is the natural one induced from $\mathcal{H}^+$ and $\Sigma\mathcal{H}^-$, and the Fock space $\mathcal{F}_P = \mathcal{F}(\mathcal{H}, \Sigma, P)$ is the completion with respect to this scalar product (indicated by the $\hat{}$ on $\otimes$). On $\mathcal{F}_P$ we define the following natural actions $a^*(z) := a^*(z^+) + a^*(z^-)$, $a(z) := a(z^+) + a(z^-)$ for $z = z^+ + z^-$, $z^+ \in \mathcal{H}^+$, $z^- \in \mathcal{H}^-$:

$$a^*(z^+)\xi \otimes \eta := (z^+ \wedge \xi) \otimes \eta \tag{3.28}$$

$$a^*(z^-)(\xi^{(n)} \otimes z'_1 \wedge \dots z'_m) := (-1)^n \xi^{(n)} \otimes \sum_{j=1}^{j=n} (-1)^{j+1} \left\langle \overline{z^-}, z'_j \right\rangle z'_1 \wedge \dots \wedge \hat{z'_j} \wedge \dots \wedge z'_m \tag{3.29}$$

$$a(z^+)(z_1 \wedge z_2 \dots \wedge z_n) \otimes \eta := \sum_{j=1}^{j=n} (-1)^{j+1} \left\langle z^+, z_j \right\rangle z_1 \wedge z_2 \dots \wedge \hat{z}_j \wedge \dots \wedge z_n \tag{3.30}$$

$$a(z^-)(\xi^{(n)} \otimes \eta := (-1)^n \xi^{(n)} \otimes \overline{z^-} \wedge \eta \tag{3.31}$$

for $\xi \in \Lambda\mathcal{H}^+$, $\xi^{(n)} \in \Lambda^n\mathcal{H}^+$, $\eta \in \Lambda\Sigma\mathcal{H}^-$ and $z_1, z_2, \dots z_n \in \mathcal{H}^+$, $z'_1, z'_2, \dots, z'_m \in \mathcal{H}^-$.

In the physics literature, a, a^* are also denoted by Ψ, Ψ^* and known as the 'field operators'. Only a (or equivalently a^*) has to be defined because of $a(z)^* = a^*(z)$.

$a : \mathcal{H} \longrightarrow \mathrm{B}(\mathcal{F}_P)$ is complex antilinear and satisfies the **canonical anticommutation relations** (CAR):

$$a(z)a(w)^* + a^*(w)a(z) = \langle z, w\rangle$$
$$a(z)a(w) + a(w)a(z) = 0$$

for $z, w \in \mathcal{H}$.

Consequently, according to theorem 3.19 there exists a unique homomorphism π_P : $\mathcal{A}(\mathcal{H}) \longrightarrow \mathrm{B}(\mathcal{F}_P)$ extending a in the sense of $a = \pi_P \circ \overline{\iota}$. π_P is a representation of the CAR algebra $\mathcal{A}(\mathcal{H})$ with a cyclic vector $\Omega = \Omega_P := 1 \in \mathcal{F}_P$: $\{\pi_P(a)\Omega_P \,:\, a \in \mathcal{A}(\mathcal{H})\,\}$ is dense in $\mathcal{F}_P$. Therefore, π_P is irreducible.

Moreover, Ω_P is uniquely defined (up to scalar multiples) as the vector $\Xi \in \mathcal{F}_P$ with

$$a(z^+)\Xi = 0\,,\;\; a^*(z^-)\Xi = 0$$

for all $z^+ \in \mathcal{H}^+$ and $z^- \in \mathcal{H}^-$.

The following theorem corresponds to the similar theorem 3.26 for Fock representations π_J of the Clifford algebra $\mathcal{C}\ell(V)$ and shows that we obtain in fact essentially different representations by varying the projections P (which is the same as varying the complex structures J of $V = \mathrm{Fix}\Sigma$).

Theorem 3.30 *Let $\mathcal{H}$ be a Hilbert space with a conjugation Σ and let P, Q be orthogonal Σ-invariant projections on $\mathcal{H}$. Then π_P and π_Q are unitarily equivalent if and only if $P - Q$ is a Hilbert Schmidt operator.*

Proof.
Under construction

□

As a consequence, in case of $P - Q$ being Hilbert Schmidt there is an intertwining unitary operator $T : \mathcal{F}_P \longrightarrow \mathcal{F}_Q$ such that $T \circ \pi_P(a) = \pi_Q(a) \circ T$ for all $a \in \mathcal{A}(\mathcal{H})$. This is a version of the result of Shale-Stinespring (cf. Theorem 0.4) and it has to do with the implementation problem: By definition, a unitary operator $U \in \mathrm{U}(\mathcal{H})$ can be *implemented* on $\mathcal{F}_P$ (or in π_P) if there exists a unitary operator $U^{\sim} : \mathcal{F}_P \longrightarrow \mathcal{F}_P$ such that

$$U^{\sim} \circ \pi_P(a) = \pi_P \theta_U(a) \circ U^{\sim}\,, \text{ for all } a \in \mathcal{A}(\mathcal{H})\,. \tag{3.32}$$

Theorem 3.31 *$U \in \mathrm{U}(\mathcal{H})$ has an implementation in π_P if and only if $[U, P - P^{\perp}]$ is Hilbert Schmidt. If it exists it is unique up to a scalar factor of modulus 1.*

This follows immediately from the preceding theorem as in the orthogonal case.

Note, that $[U, P - P^{\perp}]$ is Hilbert-Schmidt if and only if in the block matrix description of U with respect to the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the off-diagonal terms U_{+-} and U_{-+} are Hilbert-Schmidt operators.

Definition 3.32 *The restricted unitary group is*

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}, P) = \mathrm{U}_{\mathrm{res}}(\mathcal{H}) := \{U \in \mathrm{U}(\mathcal{H}) \;:\; [U, P - P^{\perp}] \text{ Hilbert Schmidt}\}\,.$$

As discussed before, the preceding theorem induces a projective representation

$$\rho_P : \mathrm{U}_{\mathrm{res}}(\mathcal{H}, P) \longrightarrow \mathrm{Aut}\,(\mathbb{P}\mathcal{F}_P)$$

which leads to a representation on the Fock space $\mathcal{F}_P$ of a central extension ${\mathrm{U}_{\mathrm{res}}}^{\sim}(\mathcal{H}, P)$ of the restricted unitary group.

ρ_P can be shown to be continuous.

$\mathrm{U}_{\mathrm{res}}(\mathcal{H}, P)$ has a natural structure of a real Banach Lie group with respect to the norm $\|U\|_r := \|U_+\| + \|U_-\| + \|U_{+-}\|_{HS} + \|U_{-+}\|_{HS}$. It has countably many connected components. The component of the identity is simply connected.

The GNS construction We compare the Fock representations of the last subsections with a class of more general representations of C^*-algebras known as GNS representations.

A *state* of a C^*-algebra A (or an *expectation value functional*) is a complex-linear form $\omega : A \longrightarrow \mathbb{C}$ satisfying $\omega(a^*a) \geq 0$ for $a \in A$ and $\omega(1) = 1$. Such a state is continuous with norm equal to 1.

For example, regarding the algebra $\mathrm{B}(\mathcal{H})$ of bounded linear operators $T : \mathcal{H} \longrightarrow \mathcal{H}$ on a Hilbert space $\mathcal{H}$ as the C^*-algebra of observables of a quantum mechanical system, each $\psi \in \mathcal{H}$ defines a state by the expectation value

$$E_\psi(T) := \frac{\langle \psi, T\psi \rangle}{\langle \psi, \psi \rangle}\, , \, T \in \mathrm{B}(\mathcal{H})\, .$$

For a CAR algebra $\mathcal{A}(\mathcal{H})$ with its embedding $\overline{\iota} : \mathcal{H} \longrightarrow \mathcal{A}(\mathcal{H})$ every self adjoint (i.e. orthogonal) projection $E \in \mathrm{B}(\mathcal{H})$ induces a state by

$$\omega_E(\overline{\iota}^*(z')\overline{\iota}(z)) := \langle z, Ez' \rangle$$

and satisfying

$$\omega_E(\overline{\iota}^*(z_1')\overline{\iota}^*(z_2') \ldots \overline{\iota}^*(z_m')\overline{\iota}(z_1) \ldots \overline{\iota}(z_m)) = \det(\langle z_i, Ez_j' \rangle)\, .$$

In general, a state ω of A induces the following representation: $\mathcal{I}_\omega := \{a \in A : \omega(a^*a) = 0\,\}$ is a closed left ideal, because $\omega(a^*b) = \overline{\omega(b^*a)}$ and $\omega(a^*b)\overline{\omega(a^*b)} \leq \omega(a^*a)\omega(b^*b)$ for $a, b \in A$. The quotient

$$\mathcal{G}_\omega := A/\mathcal{I}_\omega$$

comes with a natural positive definite Hermitian scalar product

$$\langle \psi_a, \psi_b \rangle_\omega := \omega(a^*b)$$

where $\psi_a \in \mathcal{G}_\omega$ denotes the equivalence class $\psi_a := a \bmod \mathcal{I}_\omega$. Hence, the completion of $\mathcal{G}_\omega$ is a Hilbert space which we denote with the same symbol. It comes with the natural action π_ω on $\mathcal{G}_\omega$ given by

$$\pi_\omega(a)(\psi_b) := \psi_{ab}$$

for $a, b \in A$. Of course, π_ω is a linear homomorphism (e.g. $\pi_\omega(ab)(\psi_c) = \psi_{(ab)c} = \pi_\omega(a)(\psi_{bc}) = \pi_\omega(a)\pi_\omega(b)(\psi_c)$) and a star homomorphism: $\langle \pi_\omega(a)(\psi_b), \psi_c \rangle_\omega = \langle \psi_{ab}, \psi_c \rangle_\omega = \omega((ab)^*c) = \omega(b^*(a^*c)) = \langle \psi_b, \pi_\omega(a^*)\psi_c \rangle_\omega$, hence $(\pi_\omega(a))^* = \pi_\omega(a^*)$. Note, that $\pi(a)$ is a kind of creation operator since for the vector $\xi_\omega := \psi_1$ we have $\psi_a = \pi_\omega(a)\xi_\omega$.

As a consequence $\pi_\omega : A \longrightarrow \mathrm{B}(\mathcal{G}_\omega)$ is an irreducible representation of the C^*-algebra A with cyclic vector ξ_ω. It is called the *GNS representation* for ω and ξ_ω is called the *GNS-vacuum*. The state ω can be reconstructed from $\pi = \pi_\omega$, $\xi = \xi_\omega$ as the vacuum expectation value $\omega(a) = \langle \xi, \pi(a)\xi \rangle$.

Coming back to our CAR algebra $\mathcal{A}(\mathcal{H})$ with the projection E we get the GNS representation $\pi_{\omega_E} : \mathcal{A}(\mathcal{H}) \longrightarrow \mathcal{G}_{\omega_E}$ for ω_E with cyclic vector ξ_{ω_E}.

The Fock representation produces to each orthogonal projection P another irreducible representation π_P on $\mathcal{F}_P$ with cyclic vector Ω_P. To show that π_P is unitarily equivalent to the GNS representation π_{ω_E} with $E := P^{\perp}$ we use the result that two representations of a C^*-algebra are unitarily equivalent if and only if there vacuum expectation value functionals agree.

Theorem 3.33 *Let $\mathcal{H}$ be a Hilbert space with a fixed conjugation and a conjugation invariant orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Let P denote the orthogonal projection onto $\mathcal{H}^+$ and $E = 1 - P = \Sigma P \Sigma$ the corresponding orthogonal projection onto the orthogonal complement. Then the Fock representation π_P is unitarily equivalent to the GNS representation π_{ω_E}.*

Proof. By definition $\omega_E(\overline{\iota}^*(z)\overline{\iota}(w)) = \langle w, Ez\rangle_{\mathcal{H}}$ for $z, w \in \mathcal{H}$. For the Fock representation we get $\omega_{\mathcal{F}_P}(\overline{\iota}^*(z)\overline{\iota}(w)) = \langle\, \Omega_P, a^*(z)a(w)\Omega_P\rangle_{\mathcal{F}_P} = \langle\, a(z)\Omega_P, a(w)\Omega_P\rangle_{\mathcal{F}_P} = \langle\, a(Ez)\Omega_P, a(Ew)\Omega_P\rangle_{\mathcal{F}_P} = \left\langle\, \overline{Ez}, \overline{Ew}\right\rangle_{\mathcal{H}} = \overline{\langle\, Ez, Ew\rangle_{\mathcal{H}}} = \langle\, Ew, Ez\rangle_{\mathcal{H}} = \langle\, w, Ez\rangle_{\mathcal{H}} = \omega_E(\overline{\iota}^*(z)\overline{\iota}(w))$. Hence $\omega_{\mathcal{F}_P} = \omega_E$.

$\square$